TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 278, Number 1, July 1983

TORAL ACTIONS ON 5-MANIFOLDS

BY

HAE SOO OH

ABSTRACT. We are mainly concerned with closed orientable manifolds of dimension 5 supporting effective three-dimensional torus actions. We obtain a complete classification of simply-connected manifolds of this type and a partial classification for the nonsimply-connected case.

0. Introduction. Suppose M is a closed orientable smooth manifold of dimension $(n + 2)$ with a smooth and effective T^n-action. Much work on this type of manifold has been done by Orlik and Raymond [**17**] and Raymond [**22**] in the case $n = 1$, and by Melvin [**10**], Orlik and Raymond [**18, 19**], and Pao [**21**] in the case $n = 2$. In a series of papers, we investigate this type of manifold in the case $n = 3$. This paper is the first part of the investigation.

Suppose T^3 acts on a 5-manifold M so that the set Q of singular orbits is not void. Then by the slice theorem, the orbit space M^* is a 2-manifold with $\partial M^* = Q/T^3$, as shown in Figure 1 below.

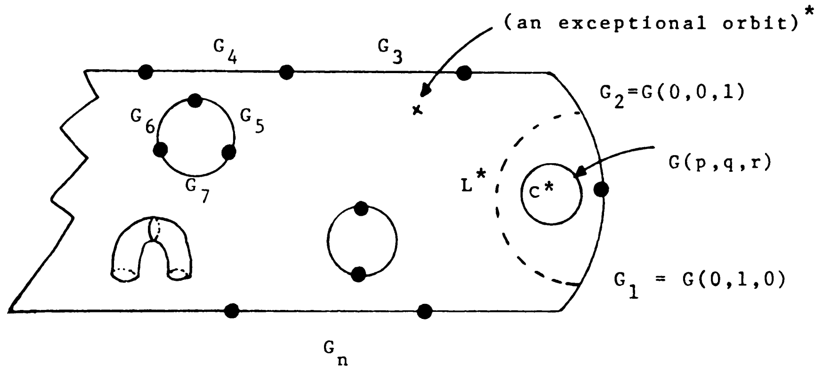


FIGURE 1

Here $G(p, q, r)$ and G_j are circle isotropy groups (defined in §1). By applying techniques similar to Orlik and Raymond [**19**], the 5-manifold M can be broken down into elementary building blocks $A(p, q, r)$, $B(a, b, c; p, q, r)$, C and D (for example, cutting M along L, which is $S^1 \times S^3$, and adding two copies of $S^1 \times D^4$

Received by the editors October 21, 1980 and, in revised form, March 15, 1982.
1980 *Mathematics Subject Classification*. Primary 57S15, 57S25.
Key words and phrases. Cross-sectioning theorem, equivariant surgery, connected sum, Stiefel-Whitney class, orbit type.

0002-9947/82/0000-1209/$05.75

equivariantly along $S^1 \times S^3$ results in two new manifolds N and $A(p, q, r)$, where N has the same orbit space as M except the boundary component c^* is deleted; thus M can be expressed as $N\#_S A(p, q, r)$, where the operation "$\#_S$", which is called an adjacent connected sum in [**9**], is elaborated in §5). These building blocks are 5-dimensional T^3-manifolds with orbit spaces as shown in Figure 2.

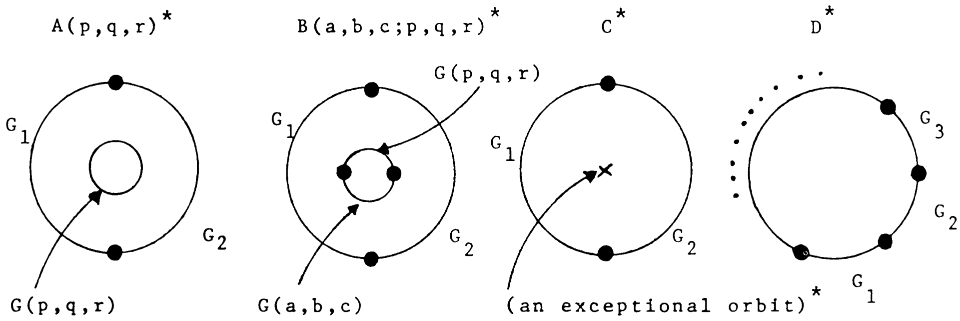


FIGURE 2

It was shown in [**15**] that $A(p, q, r)$ [$B(a, b, c; p, q, r)$] can be obtained from $\#3(S^2 \times S^3)\#(S^1 \times S^4)$ [$\#5(S^2 \times S^3)\#(S^1 \times S^4)$] or

$$(S^2 \tilde{\times} S^3)\#2(S^2 \times S^3)\#(S^1 \times S^4) \quad [(S^2 \tilde{\times} S^3)\#4(S^2 \times S^3)\#(S^1 \times S^4)]$$

by a single equivariant surgery of type (4, 2) according as $w_2 = 0$ or $w_2 \neq 0$. Here w_2 is the second Stiefel-Whitney class and $S^2 \tilde{\times} S^3$ is the nontrivial S^3-bundle over S^2. It was also discussed there when M can be expressed as a connected sum of these building blocks and some well-known manifolds. The 5-manifold D is studied in this paper.

This paper is organized as folows. In §1, we describe the orbit structure of T^n-actions on $(n+2)$-manifolds, and also prove some lemmas which are essential tools for our study. We show in §2 that $\pi_1(D)$ is a finite cycle group. We generalize this to higher dimensions. That is, if M is an $(n+2)$-dimensional T^n-manifold with orbit space a 2-disk and no exceptional orbits then $\pi_1(M)$ is a finite abelian group with at most $(n-2)$ generators.

Suppose k is the number of orbits of type T^1. Then it is shown in §§3 and 4 that D is a 5-dimensional lens space if $k = 3$ and D is a double mapping cylinder of two circle bundles over lens spaces if $k = 4$. Furthermore, if $k = 4$ and there are two equal circle isotropy groups, then D is either $S^2 \times L(p, q)$ or $S^2 \tilde{\times} L(p, q)$, the nontrivial S^2-bundle over a lens space $L(p, q)$.

Suppose that an $(n+2)$-manifold M with an effective T^n-action is simply connected and that a singular orbit exists. Then there exist no exceptional orbits [**2**, **18**] and the orbit space M^* is a 2-disk with boundary. In §5, we give examples to show that the classification theorems for simply-connected $(n+2)$-dimensional T^n-manifolds in [**8** and **9**] are not valid, and then we obtain a complete classification of simply-connected 5-manifolds with T^3-actions.

The case of $n = 3$ has some similarity to the work of Orlik and Raymond. Hence one might expect that 5-manifolds with 3, 4 and 5 orbits of type T^1, respectively, would be building blocks for the manifold D. However, we provide examples to show that this is actually not the case.

Throughout this paper, with the exception of §1, manifolds shall always mean closed connected oriented 5-manifolds with orientation-preserving T^3-actions such that orbit spaces are 2-disks with boundaries, there exist no exceptional orbits, and the isotropy groups span T^3. All actions are assumed to be smooth and effective. Unless otherwise indicated, the coefficients in all (co)homology will be the integers $\mathbf{Z}$.

This paper is a part of my doctoral dissertation written at the University of Michigan under Professor Frank Raymond. I wish to acknowledge my gratitude to him for his generous advice and assistance.

I am also grateful to the referee for his various suggestions to reorganize this paper and for suggesting short proofs of Lemmas (3.1) and (5.4).

1. Definitions and preliminary results. We shall be concerned with closed orientable smooth manifolds M of dimension $(n + 2)$ on which T^n acts smoothly and effectively, $n \geqslant 3$.

Let p: $\mathbf{R}^n \to T^n$ be the universal covering projection defined by $p(x_1, x_2, \ldots, x_n) = (e^{2\pi x_1 i}, \ldots, e^{2\pi x_n i})$. Suppose G is a circle subgroup of T^n. Then each component of $p^{-1}(G)$ is a line containing at least two lattice points. For example, the projection of a line of irrational slope in $\mathbf{R}^2$ cannot be a circle subgroup of T^2. Hence it is natural to parameterize a circle subgroup of T^n by $G(a_1, a_2, \ldots, a_n) = \{(a_1 t, a_2 t, \ldots, a_n t), \bmod \mathbf{Z}^n \mid 0 \leqslant t < 1\}$. Here $a_1, a_2, \ldots,$ and a_n are relatively prime integers. For example, we do not allow $G(0, 2, 0)$, etc. since $\gcd(0, 2, 0) = 2 > 1$.

Throughout this paper, we assume that for any subset X of M, X^* denotes its image in the orbit space M^* under the orbit map q: $M \to M^*$. Furthermore, if we are given a set X^* in M^*, we let $X = q^{-1}(X^*)$.

Let us recall that if a compact Lie group K acts smoothly on a manifold M, there exists a linear tube about each orbit, and the slice representations are faithful if the action is effective and K is abelian. We let $F = F(K, M)$ be the fixed point set of M, E the union of exceptional orbits, P the union of the principal orbits, and Q the union of the singular orbits. $K(x)$ is the orbit containing x, and K_x is the isotropy group at x.

(1.1) LEMMA. *Suppose T^n acts on an $(n + 2)$-manifold M, $n \geqslant 3$. Then we have*

(i) $F = \phi$.

(ii) *The number of orbits of type T^{n-2} is finite.*

(iii) *E^* is a finite set.*

(iv) *For any nontrivial finite subgroup H, neither $T^1 \times H$ nor $T^2 \times H$ can be an isotropy group.*

(v) *T^k, $k \geqslant 3$, cannot be an isotropy group.*

PROOF. (i) and (v) are immediate from Pak [**20**] and the slice theorem.

The linear slice of an orbit of type T^{n-2} is a 4-disk D^4, on which T^2 acts orthogonally. Hence an invariant neighborhood of an orbit of type T^{n-2} does not contain any other orbit of the same type and hence the compactness of M yields (ii).

Suppose $T^n(x)$ is an exceptional orbit. Then $(T^n)_x$ is a subgroup of $SO(2) \subset T^n$. Hence $(T^n)_x$ is a finite cyclic group $\mathbf{Z}/q$. In other words, $\mathbf{Z}/p \times \mathbf{Z}/q$ cannot be an isotropy group unless p and q are relatively prime. Again, the compactness of M and the slice theorem imply (iii).

An orbit of type $T^1 \times H$ has a slice D^3 on which $T^1 \times H$ acts orthogonally. Since the slice representation in this context is faithful, $T^1 \times H \subset SO(3)$. But $T^1 \times H$ cannot be a subgroup of SO(3) unless H is a trivial group. Similarly, an isotropy group $T^2 \times H$ can be regarded as a subgroup of SO(4). But we can easily see that SO(4) does not contain (SO(2) $\times$ SO(2)) $\times$ (a nontrivial group) as a subgroup. ■

By using the slice theorem, we can easily prove that if $Q \neq \phi$ then Q^* is the boundary of a 2-manifold M^*. Moreover, if M has an orbit of type T^{n-2}, then it has at least two orbits of type T^{n-2}. It is obvious that any two adjacent circle isotropy groups on the boundary of M^* have trivial intersection.

By the *determinant of n circle subgroups* $G(a_{11}, a_{12}, \ldots, a_{1n}), \ldots,$ and $G(a_{n1}, a_{n2}, \ldots, a_{nn})$ of T^n, we mean the determinant of the $n \times n$ matrix

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

(1.2) LEMMA. *Two circle subgroups $G(a_1, a_2, \ldots, a_n)$ and $G(b_1, b_2, \ldots, b_n)$ of T^n have trivial intersection if and only if there exist $G_i \in \mathbf{Z}^n$, $i = 3, \ldots, n$, such that the determinant of $G(a_1, a_2, \ldots, a_n)$, $G(b_1, b_2, \ldots, b_n)$, $G_3, \ldots, G_n$ is ± 1.*

(1.3) COROLLARY. $G(p, q, r) \cap G(0, 1, 0) = 1$ *if and only if* $\gcd(p, r) = 1$.

PROOF. If $\gcd(p, r) = 1$, then there exist integers x and y such that $xr - yp = 1$. Hence two circle groups have trivial intersection, since we have

$$\det\begin{pmatrix} p & 0 & x \\ q & 1 & 0 \\ r & 0 & y \end{pmatrix} = -1.$$

Conversely, if $\gcd(p, r) = d > 1$, then $p = p'd$ and $r = r'd$ for some integers p' and r'. Hence for any $(x, y, z) \in \mathbf{Z}^3$,

$$\det\begin{pmatrix} p & 0 & x \\ q & 1 & y \\ r & 0 & z \end{pmatrix} = d(p'z - xr') \neq \pm 1. \quad ■$$

Suppose the determinant of n circle subgroups of T^n, $G(a_{11}, a_{12}, \ldots, a_{1n}), \ldots,$ $G(a_{n1}, \ldots, a_{nn})$, is not zero. Then the n vectors $(a_{11}, a_{12}, \ldots, a_{1n}), \ldots, (a_{n1}, \ldots, a_{nn})$ span $\mathbf{R}^n$. Hence the n circle subgroups span T^n. The condition for n circle subgroups to be the generators of T^n is the following.

(1.4) LEMMA. *The n circle subgroups generate T^n, that is, $T^n = G(a_{11}, a_{12}, \ldots, a_{1n}) \times \cdots \times G(a_{n1}, a_{n2}, \ldots, a_{nn})$, if and only if the determinant of n circle subgroups is* ± 1.

One of the most important tools for investigating codimension two toral actions is the cross-sectioning theorem. Orlik and Raymond [**18**] showed the existence of a cross section for an orbit map in the case of $n = 2$ under the conditions $E = \varnothing$ and $Q \neq \varnothing$. But one can generalize this to the case of $n \geqslant 3$ by applying a similar technique.

(1.5) THEOREM (ORLIK AND RAYMOND). *An orbit map $M^{n+2} \to M^{n+2}/T^n$ has a cross section, provided that $Q \neq \varnothing$ and $E = \varnothing$.*

Suppose M and M' are $(n+2)$-manifolds with T^n-actions and s and s' are cross sections to each orbit map. If f^* is a weight-preserving diffeomorphism from M^* onto $(M')^*$, then there is an equivariant diffeomorphism f which covers f^*. Furthermore, by applying a similar technique to Orlik and Raymond [**19**] for exceptional orbits, we can prove an equivariant classification theorem.

(1.6) THEOREM. *Two T^n-manifolds M and M' of dimension $(n+2)$ are equivariantly diffeomorphic if and only if there is a weight perserving diffeomorphism from M^* onto $(M')^*$.*

(1.7) REMARK. Suppose T^3 acts on a 5-manifold M so that the isotropy groups span T^k, $k < 3$. Then, by (1.6), M is $N \times T^1$, where N is a 4-manifold with a T^2-action. Furthermore, if we assume that M^* is a closed 2-disk and $E = \varnothing$, then N is $S^3 \times S^1$, or a connected sum of copies of S^4, $\pm CP^2$, and $S^2 \times S^2$ (see Orlik and Raymond [**18**]).

2. Fundamental groups. Recall the orbit space of a manifold under consideration was assumed to be a 2-manifold D^* described in §0. In this section, we show that the fundamental group of a 5-manifold M $(= D)$ with a T^3-action is generated by any orbit of type T^1.

By the slice theorem, an invariant tubular neighborhood of an orbit of type T^1 is a D^4-bundle over T^1 with structure group T^2. It follows from (1.6) that this bundle is trivial (i.e. $D^4 \times T^1$).

(2.1) THEOREM. *If M is a 5-manifold supporting a T^3-action, then $\pi_1(M)$ is a finite cyclic group generated by any orbit of type T^1.*

PROOF. If α is an element of $\pi_1(M)$, then by the Whitney embedding theorem, there is an embedding $f\colon S^1 \to M$ which represents α. By the general position theorem, f is homotopic to an embedding $g\colon S^1 \to P$, the union of principal orbits. Hence $j_\#\colon \pi_1(P) = \pi_1(D^2 \times T^3) \to \pi_1(M)$ is surjective, where $j_\#$ is the homomorphism induced by inclusion.

Since it was assumed that the isotropy groups span T^3, there exist three distinct circle isotropy groups $G_1 = G(1,0,0)$, $G_2 = G(0,1,0)$ and $G_3 = G(a,b,c)$, whose determinant is not zero. Let $[G_i x]$ be the homotopy class represented by the circle $G_i x$, $x \in P$. Then $G_i x$ bounds a disk in M and hence $f_\#([G_i x]) = 1$ for $i = 1,2,3$.

We thus have

$$\pi_1(M) = \mathbf{Z}^3/(\text{kernel}\, j_{\#}) \subset \mathbf{Z}^3/\langle [G_1x], [G_2x], [G_3x] \rangle. \quad \blacksquare$$

(2.2) COROLLARY. *If there exist three distinct circle isotropy groups whose determinant is* ± 1, *then* M *is simply connected.*

PROOF. Suppose $G(a_1, b_1, c_1)$, $G(a_2, b_2, c_2)$ and $G(a_3, b_3, c_3)$ are three circle isotropy groups whose determinant is ± 1. Then it follows from (1.4) that $G(a_1, b_1, c_1) \times G(a_2, b_2, c_2) \times G(a_3, b_3, c_3)$ is isomorphic to T^3, and hence $\langle [G(a_1, b_1, c_1)x], [G(a_2, b_2, c_2)x], [G(a_3, b_3, c_3)x] \rangle$ is isomorphic to $\mathbf{Z}^3$, where $x \in P$. Therefore M is simply connected. $\blacksquare$

(2.3) COROLLARY. *If* M' *is the manifold obtained by an equivariant surgery along an orbit of type* T^1, *then* M' *is simply connected.*

PROOF. Suppose the orbit space M^* is as shown in Figure 3.

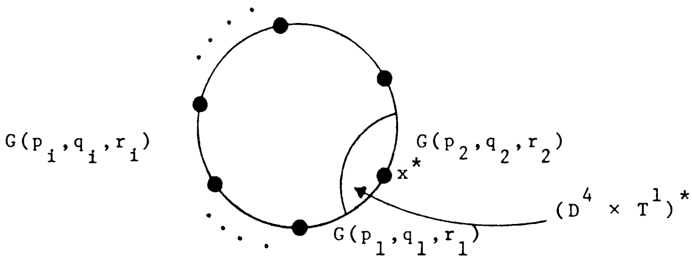


FIGURE 3

Since $G(p_1, q_1, r_1) \cap G(p_2, q_2, r_2) = 1$, we can choose relatively prime integers a, b and c so that

$$\det A = \det \begin{pmatrix} p_1 & p_2 & a \\ q_1 & q_2 & b \\ r_1 & r_2 & c \end{pmatrix} = 1.$$

Parameterize S^5 by $\{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_1\bar{z}_1 + z_2\bar{z}_2 + z_3\bar{z}_3 = 1\}$ and define a T^3-action θ on S^5 by

$$\theta((p, q, r), (z_1, z_2, z_3)) = \left(z_1 e^{2\pi\bar{p}i}, z_2 e^{2\pi\bar{q}i}, z_3 e^{2\pi\bar{r}i}\right)$$

where

$$\begin{pmatrix} \bar{p} \\ \bar{q} \\ \bar{r} \end{pmatrix} = A^{-1} \begin{pmatrix} p \\ q \\ r \end{pmatrix} \quad \text{and} \quad (p, q, r) \in T^3.$$

Then the orbit space S^5/T^3 is as shown in Figure 4.

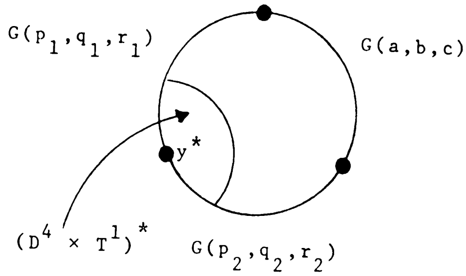


FIGURE 4

Doing an equivariant surgery along an orbit $T^3(x)$ of type T^1 is just cutting out the interiors of invariant tubular neighborhoods of $T^3(x)$ and $T^3(y)$ and pasting along their boundaries. Hence M'/T^3 is as shown in Figure 5. Since the determinant of $G(p_1, q_1, r_1)$, $G(p_2, q_2, r_2)$ and $G(a, b, c)$ is 1, by (2.2), M is simply connected. ■

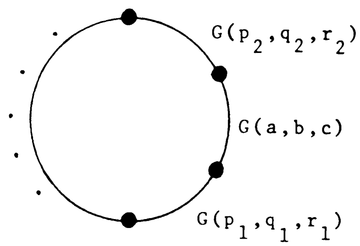


FIGURE 5

(2.4) EXAMPLE. Suppose T^3 acts on a manifold M of dimension 5 smoothly and effectively so that M^* is as shown in Figure 6. The orbit $T^3(x)$ of type T^1 is homeomorphic to $T^3/(G_1 \times G_2) \approx G(0,0,1)$. By (2.1), $\pi_1(M)$ is generated by $[T^3(x)]$ and so is finite cyclic.

$$G_1 = G(1,0,0); \qquad G_2 = G(0,1,0); \qquad G_3 = G(2,0,3);$$
$$G_4 = G(7,11,77); \qquad G_5 = G(4,5,6).$$

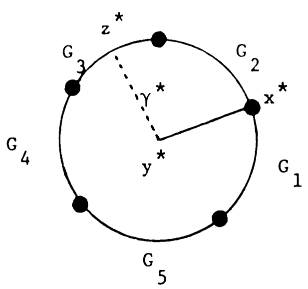


FIGURE 6

Since $(2,0,3) = 2(1,0,0) + 0(0,1,0) + 3(0,0,1)$ as vectors in $\mathbf{R}^3$, we have

$$f^y_\#([G(2,0,3)]) = [G(2,0,3)y] = 2[G(1,0,0)y] + 3[G(0,0,1)y]$$

where $f^y\colon (T^3, 1) \to (M, y)$ is the evaluation map defined by $f^y(t) = ty$.

Choose a path γ: $[0,1] \to M$ connecting y to z. $G(2,0,3)y$ is homotopic to $G(2,0,3)z = \{z\}$ by the homotopy H: $G(2,0,3) \times I \to M$ defined by

$$H(g,t) = f^{\gamma(t)}(g) = g\gamma(t).$$

Hence $[G(2,0,3)y] = 0$. By the same reason, $[G(1,0,0)y] = 0$ and hence $3[G(0,0,1)y] = 0$. This implies that the generator $[T^3(x)] = [G(0,0,1)y]$ has order dividing 3.

Similarly, the relation $(7,11,77) = 7(1,0,0) + 11(0,1,0) + 77(0,0,1)$ implies that the generator $[G(0,0,1)y]$ has order dividing 77.

Since $\gcd(3,77) = 1$, M is simply connected. But the determinant of any three isotropy groups of G_1, G_2, G_3, G_4 and G_5 is different from ± 1.

This example shows that the converse of (2.2) is not true in general. The existence of a 5-manifold M with the given orbit data is a consequence of the cross-sectioning theorem. We shall discuss it in (4.7).

(2.5) REMARK. By the techniques similar to those in this section, (2.1) and (2.2) can be extended to higher-dimensional manifolds.

(2.1)′ If M is an $(n+2)$-manifold with a T^n-action, then $\pi_1(M)$ is a finite abelian group with at most $(n-2)$ generators.

(2.2)′ If there exist n-distinct circle isotropy groups whose determinant is ± 1, then M is simply connected.

3. 5-manifolds with three orbits of type T^1. It was assumed that the isotropy groups span T^3. Thus the number of orbits of type T^1 is at least three. In this section, we show that a 5-manifold with three orbits of type T^1 is a 5-dimensional lens space $L(p,q,r)$.

(3.1) LEMMA. *A 5-dimensional lens space $L(p,q,r)$ admits a T^3-action with orbit space*

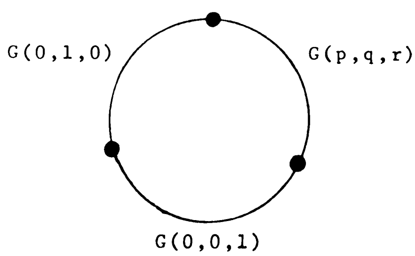


FIGURE 7

PROOF. Define a T^3-action on S^5 by the matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ q & 1 & 0 \\ r & 0 & 1 \end{pmatrix}^{-1}$$

(see 2.3), and set $K = \{(j/p, 0, 0) \mid 0 \leq j < p\} \subset T^3$. Factoring S^5 by the K-action inside the T^3-action, we obtain a 5-dimensional lens space $L(p, -q, -r)$

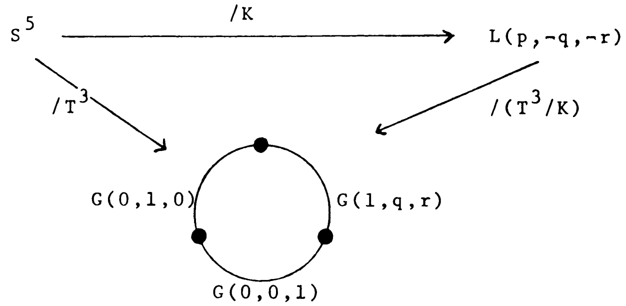


FIGURE 8

Now the isomorphism $T^3/K \to T^3$ defined by $(x, y, z) + K \to (px, y, z)$ gives rise to an action of T^3 on $L(p, -q, -r)$ with orbit space as shown in Figure 7. Since $L(p, -q, -r)$ is homeomorphic to $L(p, q, r)$ we have the conclusion. ■

Suppose a 5-manifold M has a T^3-action with orbit space

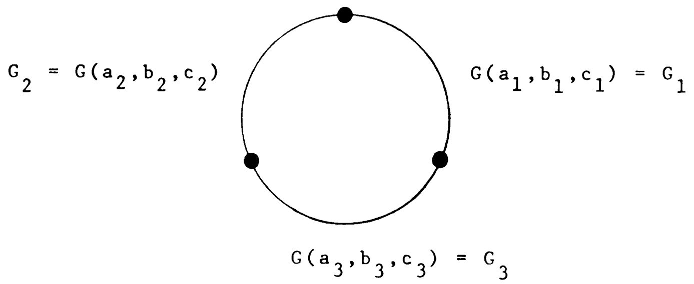


FIGURE 9

Then there exists an automorphism f of T^3 which maps G_1, G_2 and G_3 to $G(p, q, r)$, $G(0, 1, 0)$ and $G(0, 0, 1)$, respectively. Hence (3.1) and (1.6) yield the following.

(3.2) THEOREM. *A 5-manifold M with three orbits of type T^1 is a lens space $L(p, q, r)$, where p, q, r are the integers described above.*

4. 5-manifolds with four orbits of type T^1. In this section, we determine 5-manifolds with four orbits of type T^1. Reparameterizing (if necessary), we may assume the orbit space of this manifold is as shown in Figure 10.

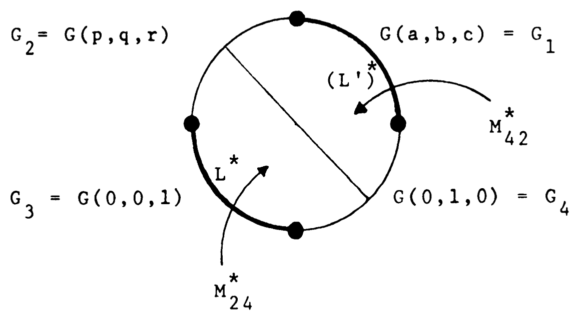


FIGURE 10

(4.1) LEMMA. *The space L sitting over L^* is the lens space $L(p, q)$ (notice: $p = -\det(G_2, G_3, G_4)$) and the space M_{24} sitting over M_{24}^* is a D^2-bundle over L with structure group* SO(2). *Hence M is a double mapping cylinder over two circle bundles over lens spaces L and L'.*

PROOF. Let $L = q^{-1}(L^*)$, where q is the orbit map. Then L is a 3-manifold with a noneffective T^3-action and L/T^3 is

$$G_2 \times G_3 \bullet\!\!\underset{G_3}{\longrightarrow}\!\!\bullet\ G_3 \times G_4.$$

Factoring T^3 by G_3, L is also a T^2-manifold with orbit space

$$G(p, q) \bullet\!\!\longrightarrow\!\!\bullet\ G(0, 1).$$

Hence it follows from [**18**] that L is a lens space $L(p, q)$.

Since the action was assumed to be smooth, it is possible to choose a Riemannian metric so that T^3 acts as a group of isometries. By Bredon [**2**, VI, 2.2], M_{24} is an invariant tubular neighborhood of L. Now $G(0, 0, 1)$ acts on the normal bundle freely away from L. Thus M_{24} is a 2-disk bundle over L with structure group G_3. ■

By using the techniques similar to those in (2.4) we have the following.

(4.2) COROLLARY. *Suppose d_i is the determinant of the adjacent triple G_i, G_{i+1}, G_{i+2}, and n is the number of orbits of type T^1. If $\pi_1(M) = \mathbf{Z}/m$, then* $\gcd(d_1, \ldots, d_n) \geqslant m$ *and if* $\gcd(d_1, \ldots, d_n) = 1$, *then M is simply connected.*

Suppose the number of orbits of type T^1 is four. Then ∂M^* is divided into four arcs by the orbits of type T^1. We may assume that the orbit data are as in Figure 11.

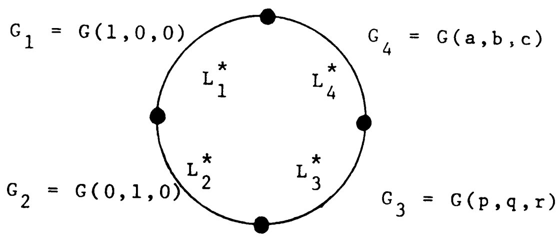


FIGURE 11

If L_2 is $S^1 \times S^2$, then by (4.1), $\det(G_1, G_2, G_3) = r = 0$ and hence by (1.3), p should be ± 1. Since it was assumed that the isotropy groups span T^3, c must not be zero. If L_1 is also $S^1 \times S^2$ then by (4.1), $\det(G_1, G_2, G_4) = c$ is zero which is a contradiction. By a similar reason, L_3 cannot be $S^1 \times S^3$. Thus we have

(4.3) LEMMA. *If one of four arcs is $(S^1 \times S^2)^*$, then neither of the two arcs next to this is $(S^1 \times S^2)^*$.*

(4.4) LEMMA. (i) *There exist two equal circle isotropy groups if and only if two arcs out of four correspond to $S^1 \times S^2$.*

(ii) *If there are two equal circle isotropy groups, then in fact M is an S^2-bundle over a lens space ($\neq S^1 \times S^2$) with structure group* SO(2).

PROOF. (i) Suppose two arcs correspond to $S^1 \times S^2$. Then it follows from (4.3) that either L_1 and L_3 or L_2 and L_4 are $S^1 \times S^2$. Say L_2 and L_4 are $S^1 \times S^2$. Then the orbit space M^* is as shown in Figure 12.

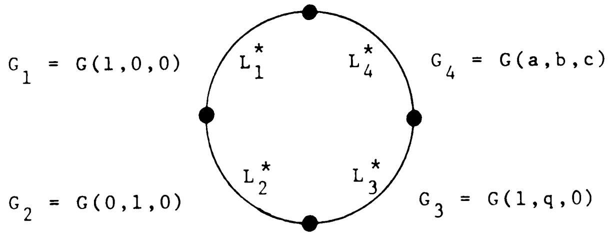


FIGURE 12

Since L_4 is $S^1 \times S^2$, by (4.1), $\det(G_3, G_4, G_1) = qc = 0$. Since the isotropy groups span T^3, c is not zero and hence $q = 0$. So $G_1 = G(1,0,0) = G_3$. Conversely, if $G_1 = G_3$, then $\det(G_1, G_2, G_3) = 0$ and $\det(G_3, G_4, G_1) = 0$. Hence L_2 and L_4 are $S^1 \times S^2$.

(ii) If G_1 is equal to G_3 then by (i), L_2 and L_4 are $S^1 \times S^2$ and hence by (4.3), neither L_1 nor L_3 is $S^1 \times S^2$. By (4.1), L_1 is a lens space $L(m,n)$ and L_2 is $-L(m,n)$. Hence M is an S^2-bundle over $L(m,n)$, $m \neq 0$. ■

Let ξ be an S^2-bundle over a lens space X with structure group SO(2). Then we may consider ξ as an SO(3)-bundle. Since the Euler class of ξ is zero and $w_1(\xi) \in H^1(X; \mathbf{Z}/2)$ is 0, $w_2(\xi)$, the second Stiefel-Whitney class, is the only obstruction class. Since $\pi_i(B\,\mathrm{spin}(3)) = \pi_{i-1}(\mathrm{spin}(3))$ and $\pi_{i-1}(\mathrm{spin}(3)) = \pi_{i-1}(S^3) = 0$ for $i \leq 3$, we have $[X, B\,\mathrm{spin}(3)] = 0$, since $\dim X \leq 3$. It is known that ξ admits a spin structure if and only if $w_2(\xi) = 0$.

Since $\dim X \leq 3$, by attaching cells to BSO(3), we can construct a $K(\mathbf{Z}/2, 2)$-space such that $[X, \mathrm{BSO}(3)] \to [X, K(\mathbf{Z}/2, 2)]$ is a bijection. That is, we have $[X, \mathrm{BSO}(3)] \approx [X, K(\mathbf{Z}/2, 2)] = H^2(X; \mathbf{Z}/2) = 0$ or $\mathbf{Z}/2$. Hence there is at most one nontrivial S^2-bundle over X. We claim that ξ is not always trivial. In the Bockstein sequence associated with

$$0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}/2 \to 0,$$

$$H^2(X; \mathbf{Z}) \to H^2(X; \mathbf{Z}) \to H^2(X; \mathbf{Z}/2) \to H^3(X; \mathbf{Z}) \xrightarrow{\alpha} H^3(X; \mathbf{Z}),$$

the homomorphism α is injective. So if $w_2(\xi) \neq 0$, then there is an $f \in H^2(X; \mathbf{Z})$ such that $f/(\mathbf{Z}/2) = w_2(\xi) \neq 0$. In other words, ξ has a nonzero chern class and hence ξ is nontrivial. Thus we have

(4.5) THEOREM. *If a 5-manifold M has a T^3-action such that the number of orbits of type T^1 is four and there are two equal circle isotropy groups, then M is either $S^2 \times L(m,n)$ or $S^2 \tilde{\times} L(m,n)$. Here $S^2 \tilde{\times} L(m,n)$ is the nontrivial S^2-bundle over the lens space $L(m,n)$ and m, n are integers described in* (4.1) *and* (4.4).

5-manifolds with T^3-actions have some similarities to those studied by Orlik and Raymond [**18**, **19**] in dimension 4. Hence, one might expect that the 5-manifolds with 3, 4 and 5 orbits of type T^1, respectively, could be elementary building blocks for a 5-manifold with orbit space D^* from §0. But the following example shows that this is not the case.

(4.6) EXAMPLE. Suppose M is a T^3-manifold of dimension 5 with orbit space as shown in Figure 14. Then M is simply connected and any two nonadjacent isotropy groups have nontrivial intersection. Hence, even if M is simply connected, M cannot be broken down into simpler pieces.

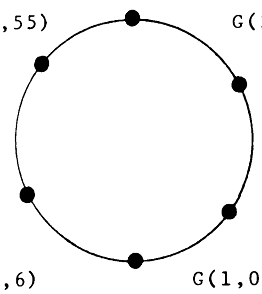


FIGURE 13

PROOF. Since $\gcd(55, 3) = 1$, by an argument similar to (2.4), M is simply connected. By (1.2), any two circle isotropy groups $G(p, q, r)$ and $G(p', q', r')$ have trivial intersection if and only if there are relatively prime integers x, y, z such that

$$\det\begin{pmatrix} p & p' & x \\ q & q' & y \\ r & r' & z \end{pmatrix} = 1.$$

This condition is equivalent to $\gcd(qr' - rq', pr' - rp', pq' - qp') = 1$. However we can see that $\gcd(gr' - rq', pr' - rp', pq' - qp') > 1$ unless $G(p, q, r)$ and $G(p', q', r')$ are adjacent to each other. Hence any arc connecting two points which are not in adjacent arcs in M^* does not correspond to $S^1 \times S^3$ in M. ■

The orbit data of (4.6) are legally weighted (that is, any two adjacent circle isotropy groups have trivial intersection). Hence, the following remark implies the existence of a T^3-manifold M with the given orbit data.

(4.7) REMARK. In [7], a simply-connected manifold of dimension $(n + 2)$, supporting an effective T^n-action, was constructed. Their construction is beautiful but complicated. We now give a simpler construction of this type of manifold. Let $T^n = G(1, 1, \dots, 0) \times G(0, 1, 0, \dots, 0) \times \cdots \times G(0, 0, \dots, 0, 1)$ and D^2 be a closed 2-disk. Divide ∂D^2 into n arcs $L_1^*, L_2^*, \dots, L_n^*$. Let M be the quotient space $(T^n \times D^2)/\sim$ by the relation "$\sim$" on $T^n \times D^2$ defined by

$$\begin{cases} (g, x) \sim (0, x) & \text{if } (g, x) \in G_i \times L_i^*, \\ (g, x) \sim (0, x) & \text{for } g \in G_i \times G_{i+1} \text{ if } \{x\} = L_i^* \cap L_{i+1}^*, \\ (g, x) \sim (g, x) & \text{otherwise, where } G_i = G(0, \dots, 0, 1^i, 0, \dots, 0). \end{cases}$$

Then M admits an effective T^n-action in a natural way and the orbit space M^* with respect to the action is shown in Figure 14. (1.6) can be applied to show that M is actually an $(n + 2)$-manifold. Moreover, (2.5) yields the simple connectivity of M.

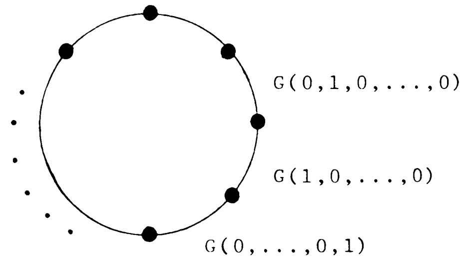


FIGURE 14

By using techniques similar to this, one can show that for any legally weighted orientable 2-manifold M^* there is an $(n + 2)$-dimensional T^n-manifold M with orbit space M^*.

5. Simply-connected 5-manifolds with T^3-actions. McGavran claimed to have classified simply-connected $(n + 2)$-manifolds supporting effective T^n-action, for $n = 3$ and 4 [**8**] and for $n \geqslant 5$ in [**9**]. In this section, we shall give examples to show that there are some gaps in his claims and then we give a complete classification theorem for simply-connected 5-manifolds with T^3-actions. A classification theorem for simply-connected 6-manifolds with T^4-actions was proved in [**16**].

The manifolds in question do not, in general, satisfy his lemma ((2.1) of [**8**]). Even if one restricts to manifolds nice enough to satisfy his lemma, it will, in general, be impossible to obtain his conclusions for several reasons. First, the simple-connectivity is not inherited in the inductive step. Secondly, even if it is inherited, an equivariant replacement may create, contrary to his claims, a manifold with nonvanishing second Stiefel-Whitney class.

For the sake of completeness we prove the following.

(5.1) LEMMA. *Suppose the second Stiefel-Whitney class of a* 5*-manifold M is not zero and let M' be the manifold resulting from a surgery of type* (2, 4) *in the sense of Milnor* [**12** *or* **13**]. *Then $w_2(M')$ is also not zero.*

PROOF. Let W be the trace of the surgery. Then it follows the duality that

$$H^2(W, M') = H_4(W, M) = H_4(M \cup \text{2-cell}, M) = 0.$$

Hence the cohomology exact sequence for (W, M') gives rise to the injectivity of j^*: $H^2(W) \to H^2(M')$. Since $w_2(W) \neq 0$, by the naturality of Stiefel-Whitney class, $w_2(M') = j^*(w_2(W)) \neq 0$. ■

The classification theorem for simply-connected 5-dimensional T^3-manifolds in [**8**] does not include any manifolds with nonvanishing second Stiefel-Whitney class. Contrary to this we have the following.

(5.2) THEOREM. *There exist simply-connected 5-dimensional T^3-manifolds with nonvanishing second Stiefel-Whitney classes.*

PROOF. From the definition of the 5-dimensional lens space $L(2k, 1, 1)$, we have a commutative diagram

$$\begin{array}{ccc} S^5 & \xrightarrow{/(\mathbf{Z}/2)} & \mathbf{R}P^5 = L(2,1,1) \\ & \underset{/(\mathbf{Z}/2k)}{\searrow} & \big\downarrow /(\mathbf{Z}/k) \\ & & L(2k,1,1) \end{array}$$

Since $w_2(\mathbf{R}P^5) \neq 0$, we have $w_2(L(2k, 1, 1)) \neq 0$. It follows from (3.1) that $L(2k, 1, 1)$ has a T^3-action with orbit space shown in Figure 15. Suppose M' is the manifold resulting from equivariant surgery along an orbit $T^3(x)$ of type T^1. Then M^1 has three circle isotropy groups whose determinant is ± 1 (see the proof of (2.3)). Hence by (2.3), M' is simply connected and by (5.1), M' has nonvanishing second Stiefel-Whitney class. ■

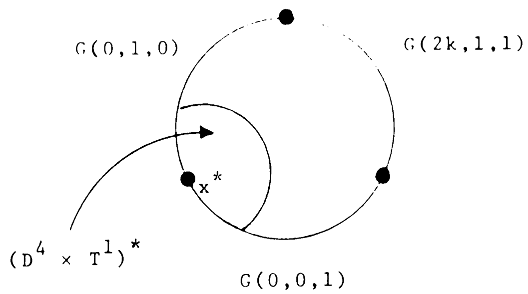


FIGURE 15

Suppose M_1 and M_2 are 5-manifolds. Then the connected sum $M_1 \# M_2$ is obtained by removing the interior of a 5-ball from each and attaching the resulting manifolds together along their boundaries. By modifying Goldstein-Lininger's construction on manifolds **[6]**, we now introduce another construction as follows. Suppose M_1 and M_2 are T^3-manifolds. The interior of an invariant tubular neighborhood $S^1 \times D^4$ of an orbit of type T^1 can be removed from M_1 and M_2. The resulting manifolds can be attached equivariantly along $S^1 \times S^3$ to obtain a *new manifold* $M_1 \#_S M_2$ *with a* T^3-*action*.

Now we give examples which show that the crucial lemma [**8**, Lemma 2.1] is not true.

(5.3) COUNTEREXAMPLES. (i) By (3.1), lens spaces $L(4, 1, 1)$ and $L(7, 1, 1)$ have T^3-actions with respective orbit spaces shown in Figure 16. Let $M = L(4, 1, 1) \#_S L(7, 1, 1)$. Then the orbit space M^* is as shown in Figure 17. Let $N_1 = L(4, 1, 1) - \mathrm{int}(D^4 \times T^1)$ and $N_2 = L(7, 1, 1) - \mathrm{int}(D^4 \times T^1)$. Then $\pi_1(N_1) = \{a \mid a^4 = 1\}$ and $\pi_1(N_2) = \{b \mid b^7 = 1\}$. Hence by the Van Kampen Theorem, M is simply connected. Since the determinants of adjacent triples are 3, –4, 3 and 7, it follows from (1.4) that no adjacent triples generate T^3.

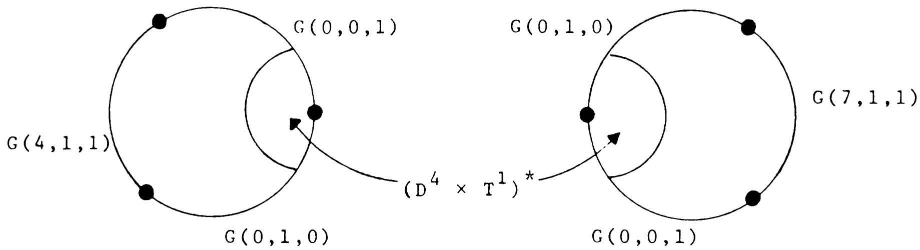


FIGURE 16

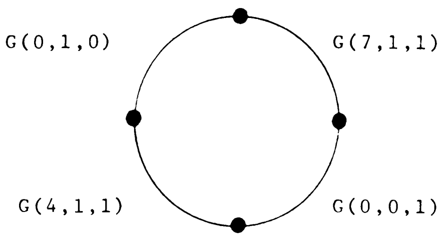


FIGURE 17

(ii) Suppose M is a 6-dimensional T^4-manifold with orbit space, shown in Figure 18, where $G_1 = G(1,0,0,0)$, $G_2 = G(0,1,0,0)$, $G_3 = G(2,1,6,9)$, $G_4 = G(0,1,0,0)$, $G_5 = G(0,0,0,1)$, $G_6 = G(6,3,2,4)$, $G_7 = G(0,0,1,0)$, $G_8 = G(0,1,0,0)$ and $G_9 = G(6,3,11,44)$. Then M is simply connected, since $\det(G_1, G_2, G_5, G_7) = 1$. By (1.4), G_i, G_{i+1}, G_{i+2} generate T^3 if and only if there exists an element X of $\mathbf{Z}^4$ such that $\det(G_i, G_{i+1}, G_{i+2}, X) = \pm 1$. But $\det(G_i, G_{i+1}, G_{i+2}, X) \neq \pm 1$ for any $X \in \mathbf{Z}^4$ and hence there is no adjacent triple which generates T^3. ■

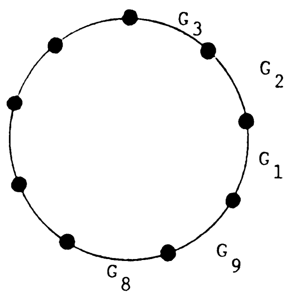


FIGURE 18

In his classification of simply-connected 5-manifolds [**1**], Barden showed that $H_2(M)$ and $i(M)$ determined by $w_2(M)$ form a complete set of invariants for the diffeomorphism classification. Now we are going to identify simply-connected 5-manifolds with T^3-actions by using Barden's classification theorem. Hence we need to compute $H_2(M)$.

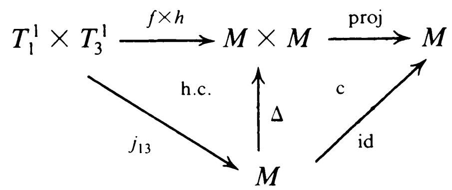
(5.4) LEMMA. *Let M be a T^3-manifold of dimension 5 and suppose k is the number of orbits of type T^1. Then $H_2(M) \approx \mathbf{Z}^{k-3}$.*

PROOF. Recall $P = q^{-1}(\text{int } M^*)$ and $Q = q^{-1}(\partial M^*)$, where q is the orbit map $M \to M^*$. Then $P \approx D^2 \times T^3$. Parameterize $T^3 = T_1^1 \times T_2^1 \times T_3^1$ by $\{(x, y, z) \mid 0 \leqslant x, y, z \leqslant 1\}$. Suppose that j denotes the inclusion map from T^3 into M and that j_{12}, j_{13}, j_{23}, f, g and h denote the restrictions of j to $T_1^1 \times T_2^1$, $T_1^1 \times T_3^1$, $T_2^1 \times T_3^1$, T_1^1, T_2^1 and T_3^1, respectively.

By the Künneth formula, $H_2(T^3) = H_1(T_1^1) \otimes H_1(T_2^1) + H_1(T_1^1) \otimes H_1(T_3^1) + H_1(T_2^1) \otimes H_1(T_3^1)$. Since the isotropy groups span T^3, we can assume that $T_1^1 = G(1,0,0)$ and $T_2^1 = G(0,1,0)$ are isotropy groups and hence, by an argument similar to that of (2.4), $f_*\colon H_1(T_1^1) \to H_1(M)$ and $g_*\colon H_1(T_2^1) \to H_1(M)$ are zero-maps. Since the Künneth formula is functorial, we have a commutative diagram

$$\begin{array}{ccc} H_1(T_1^1) \otimes H_1(T_3^1) & \overset{\times}{\underset{\approx}{\to}} & H_2(T_1^1 \times T_3^1) \\ f_* \otimes h_* \downarrow & & \downarrow (f \times h)_* \\ H_1(M) \otimes H_1(M) & \overset{\times}{\to} & H_2(M \times M) \end{array}$$

Since f_* is the zero-map, so is $(f \times h)_*$. On the other hand, the left half of the following diagram is homotopy commutative by a homotopy defined by $H(x, y, t) = j_{13}(x, t + (1-t)y) \times j_{13}(t + (1-t)x, y)$.



Here Δ is the diagonal map. Since $\Delta_*(j_{13})_* = (f \times h)_*$ is the zero-map and Δ_* is injective, $(j_{13})_*$ is also the zero-map. Similarly, $(j_{12})_*$ and $(j_{23})_*$ are zero-maps and hence $j_*\colon H_2(P) \to H_2(M)$ is also the zero-map.

By Poincaré duality, $H_2(M, P) \approx H^3(Q)$. By an argument similar to that of (4.1), Q is a chain of several lens spaces. Hence by the Mayer-Vietoris sequence, we have $H^3(Q) \approx \mathbf{Z}^k$. Thus the homology sequence of the pair (M, P) yields

$$\begin{array}{ccccccccc} H_2(P) & \overset{j_*}{\to} & H_2(M) & \to & H_2(M, P) & \to & H_1(P) & \to & H_1(M) \\ & \| & & & \wr\wr & & \wr\wr & & \wr\wr \\ & \text{0-map} & & & \mathbf{Z}^k & & \mathbf{Z}^3 & & \mathbf{Z}/m \end{array}$$

and the result follows. ■

The S^3-bundles over S^2 with structure group SO(4) are classified by $\pi_1(\mathrm{SO}(4)) = \mathbf{Z}/2$. Suppose $S^2 \tilde{\times} S^3$ denotes the nontrivial S^3-bundle over S^2. Then $S^2 \tilde{\times} S^3$ also admits a T^3-action. In fact, in the proof of (5.2), by performing an equivariant surgery on $L(2,1,1)$, we constructed a T^3-manifold M' with orbit space shown in Figure 19. It follows from (2.2), (5.1) and (5.4) that $\pi_1(M') = 1$, $w_2(M') \neq 0$ and $H_2(M') = \mathbf{Z}$. By [**1**, (1.1) and (2.3)], M' must be the nontrivial S^3-bundle over S^2. Hence $S^2 \tilde{\times} S^3$ admits a T^3-action.

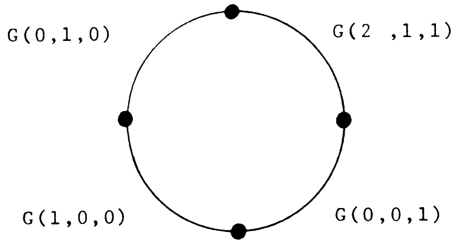


FIGURE 19

(5.5) THEOREM. *Suppose M is a simply-connected 5-manifold with a T^3-action and the number of orbits of type T^1 is k. Then we have*

$$M \approx S^5 \quad \text{if } k = 3,$$
$$M \approx \#(k-3)(S^2 \times S^3) \quad \text{if } w_2(M) = 0,$$
$$M \approx (S^2 \tilde{\times} S^3)\#(k-4)(S^2 \times S^3) \quad \text{if } w_2(M) \neq 0.$$

PROOF. (3.2), (5.4) and [**1**, (1.1) and (2.3)] yield the results. ■

(5.6) REMARK. (1) For any integer $k \geqslant 3$, $(S^2 \tilde{\times} S^3)\#(k-4)(S^2 \times S^3)$ and $\#(k-3)(S^2 \times S^3)$ actually admit T^3-actions (that is, every simply-connected 5-manifold M with $H_2(M)$ torsion free admits a T^3-action).

PROOF. (a) Let $\tilde{M}_k$ be the manifold obtained by performing successively an equivariant surgery of type (2, 4) on $L(2, 1, 1)$ $(k-3)$ times, this is,

$$\tilde{M}_k = L(2,1,1) \underbrace{\#_S S^5 \#_S S^5 \#_S \cdots \#_S S^5}_{(k-3)\text{ copies}}.$$

Then $\tilde{M}_k$ has k orbits of type T^1. Moreover, it follows from (2.3) and (5.1) that $\tilde{M}_k$ is a simply-connected T^3-manifold with nonvanishing second Stiefel-Whitney class. Thus we have $\tilde{M}_k = (S^2 \tilde{\times} S^3)\#(k-4)(S^2 \times S^3)$.

(b) It follows from [**18**] that the lens space $L(2, 1)$ admits a T^2-action so that its orbit space is

$$G(2,1) \bullet\!\!-\!\!\!-\!\!\bullet\, G(0,1).$$

Hence $S^2 \times L(2,1)$ admits a T^3-action with orbit space as shown in Figure 20. Similarly, $S^2 \times S^3$ $(= M_4)$ also admits a T^3-action so that M_4/T^3 is as shown in Figure 21.

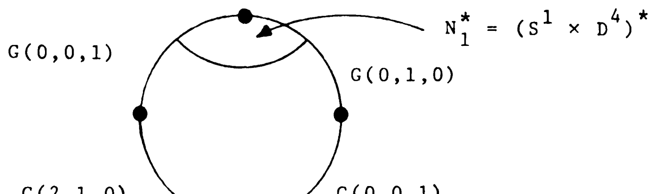


FIGURE 20

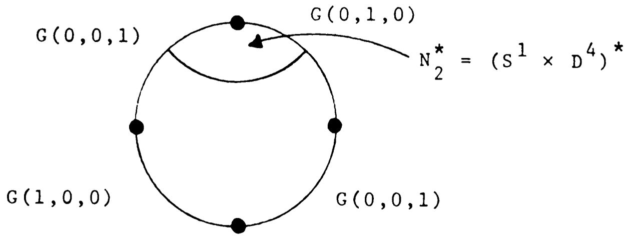


FIGURE 21

Let M_6 be the manifold constructed by gluing $P = (S^2 \times L(2,1)) - \operatorname{int} N_1$ and $Q = (S^2 \times S^3) - \operatorname{int} N_2$ equivariantly along their boundaries (this is, $M_6 = (S^2 \times L(2,1)) \#_S (S^2 \times S^3)$). Then it follows from (2.2) that M_6 is simply connected. Furthermore, $w_2(M_6)$ is zero. In fact, by the Mayer-Vietoris sequence of P and Q with $\mathbf{Z}/2$ coefficients, we have an exact sequence

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/2 \to H^2(M_6) \xrightarrow{k^*} H^2(P) \oplus H^2(Q) \to$$

Thus k^* is injective and hence $k^*(w_2(M_6)) = w_2(P) + w_2(Q) = 0$ implies $w_2(M_6) = 0$. Since M_6 is a T^3-manifold with 6 orbits of type T^1, M_6 is $\#3(S^2 \times S^3)$.

By applying a similar argument, $(S^2 \times L(2,1)) \#_S S^5$ ($= M_5$, the manifold resulting from equivariant surgery on $S^2 \times L(2,1)$ along an orbit of type T^1) is a simply-connected T^3-manifold with $w_2(M_5) = 0$ and with 5 orbits of type T^1. Hence M_5 is $\#2(S^2 \times S^3)$.

For $k \geqslant 7$, a simply-connected T^3-manifold with $w_2(M_k) = 0$ and k orbits of type T^1 can be constructed inductively by the algorithm

$$M_{2i-1} \#_S (S^2 \times L(2,1)) = M_{2i+1},$$
$$M_{2i-2} \#_S (S^2 \times L(2,1)) = M_{2i}, \quad \text{where } i \geqslant 3.$$

(2) It is worthwhile to note that the connected sums in (5.5) are not equivariant. Pak [20] showed that if T^n acts on an $(n+1)$-manifold M, then M is diffeomorphic to either T^{n+1} or $L(p,q) \times T^{n-2}$ for $n \geqslant 3$. Hence if M is an $(n+2)$-dimensional T^n-manifold, $n \geqslant 3$, then M cannot have an equivariant connected sum decomposition.

(5.7) COROLLARY. *If M is a 5-manifold with a T^3-action, then M bounds a 6-manifold.*

PROOF. Suppose M' is the manifold resulting from equivariant surgery on M along an orbit of type T^1. Then by (2.3), M' is a simply-connected 5-manifold with a T^3-action. Hence it follows from (5.5) that the Stiefel-Whitney numbers are all zero. Since the Stiefel-Whitney numbers are cobordism invariants, the Stiefel-Whitney numbers of M are all zero and hence M bounds a 6-manifold. ∎

In concluding, we generalize (5.2) to higher-dimensional manifolds. Contrary to the results in [9], we have the following.

(5.8) REMARK. There exist simply-connected $(n+2)$-manifolds with T^n-actions and nonvanishing second Stiefel-Whitney classes.

PROOF. Suppose the statement is true for $n = k$ and suppose M^{k+2} is a manifold of this type.

Let $M = M^{k+2} \times S^1$ and define a T^{k+1}-action on M by product. By the Cartan formula, $w_2(M) = w_2(M^{k+2}) \times w_0(S^1) \neq 0$. Let $N = M - \mathrm{int}(D^4 \times T^{k-1})$ and j: $N \to M$ be the inclusion map. By the universal coefficient theorem, $w_2(M)$: $H_2(M; \mathbf{Z}/2) \to \mathbf{Z}/2$ and this map is nontrivial.

By the homology sequence of the pair (M, N),

$$\begin{array}{ccccccc} \to & H_2(N;\mathbf{Z}_2) & \xrightarrow{j_*} & H_2(M;\mathbf{Z}_2) & \longrightarrow & H_2(M, N;\mathbf{Z}_2) \to \\ & & \searrow^{w_2(N)} & \downarrow{\scriptstyle w_2(M)} & & \wr\wr \\ & & & \mathbf{Z}_2 & & H_2(D^4 \times T^{k-1}, S^3 \times T^{k-1}; \mathbf{Z}_2) \\ & & & & & \wr\wr \\ & & & & & 0 \end{array}$$

we have $w_2(N) = w_2(M) \cdot j_*$ and j_* is surjective. Hence, $w_2(M) \neq 0$ yields $w_2(N) \neq 0$.

We may assume that the orbit space of M is as shown in Figure 22.

$$G_1 = G(1,0,0,\ldots,0), \quad G_2 = G(0,1,0,\ldots,0),$$
$$G_3 = G(a_{31}, a_{32}, \ldots, a_{3k}, 0), \ldots, \ G_m = G(a_{m1}, a_{m2}, \ldots, a_{mk}, 0).$$

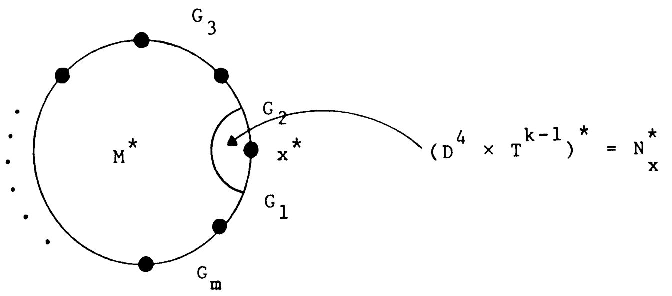


FIGURE 22

Suppose M_1 is a $(k + 3)$-dimensional T^{k+1}-manifold with orbit space, shown in Figure 23. (Existence of such a manifold was shown in (4.7).)

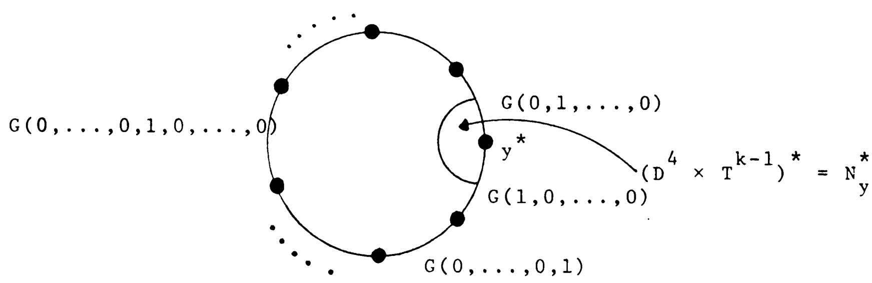


FIGURE 23

Cutting out the interiors of N_x and N_y from M and M_1 respectively and gluing the resulting spaces together along their boundaries, we obtain a $(k+3)$-manifold $\overline{M}$ with a T^{k+1}-action. By (2.5), $\overline{M}$ is simply connected and $w_2(\overline{M}) \neq 0$, since $w_2(N) \neq 0$.

REFERENCES

1. D. Barden, *Simply connected five manifolds*, Ann. of Math. (2) **82** (1965), 365–385.
2. G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.
3. W. Browder, *Surgery on simply connected manifolds*, Springer-Verlag, Berlin and New York, 1972.
4. P. Conner and F. Raymond, *Actions of compact Lie groups on aspherical manifolds*, Topology of Manifolds, Markham, Chicago, Ill., 1970, pp. 365–385.
5. A. Borel, *Seminar on transformation groups*, Princeton Univ. Press, Princeton, N.J., 1960.
6. R. Goldstein and L. Lininger, *A classification of* 6*-manifolds with free* S^1*-actions*, Lecture Notes in Math., Vol. 298, Springer-Verlag, Berlin and New York, 1971, pp. 316–323.
7. S. K. Kim, D. McGavran and J. Pak, *Torus action on simply connected manifolds*, Pacific J. Math **53** (1974), 435–444.
8. D. McGavran, T^n*-actions on simply connected* $(n+2)$*-manifolds*, Pacific J. Math. **71** (1977), 487–496.
9. ______, *Adjacent connected sums and torus actions*, Trans. Amer. Math. Soc. **251** (1979), 235–253.
10. P. Melvin, *On* 4*-manifolds with singular torus actions*, Math. Ann. **256** (1981), 255–276.
11. J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton Univ. Press, Princeton, N.J., 1974.
12. J. W. Milnor, *A procedure for killing the homotopy groups of differentiable manifolds*, Proc. Sympos. Pure Math., vol. 3, Amer. Math. Soc., Providence, R.I., 1961, pp. 39–55.
13. ______, *Lectures on the h-cobordism theorem*, Notes, Princeton Univ. Press, Princeton, N.J., 1963.
14. H. S. Oh, *Toral actions on* 5*- and* 6*-manifolds*, Ph.D. Thesis, Univ. of Michigan, 1980.
15. ______, *Toral actions on* 5*-manifolds*. II, Indiana Univ. Math. J. **32** (1983), 129–142.
16. ______, T^4*-actions on* 6*-manifolds*, Topology Appl. **13** (1982), 137–154.
17. P. Orlik and F. Raymond, *Actions of* SO(2) *on manifolds* (Proc. Conf. Transf. Groups, New Orleans, 1967), Springer-Verlag, Berlin and New York, 1968.
18. ______, *Actions of torus on* 4*-manifolds*. I, Trans. Amer. Math. Soc. **152** (1970), 531–559.
19. ______, *Actions of torus on* 4*-manifolds*. II, Topology **13** (1974), 89–112.
20. J. Pak, *Actions of torus* T^n *on* $(n+1)$*-manifolds*, Pacific J. Math. **44** (1973), 671–674.
21. P. Pao, *The topological structure of* 4*-manifolds with effective torus actions*, Ph.D. Thesis, Univ. of Michigan, 1975.
22. F. Raymond, *Classification of the actions of the circle on* 3*-manifolds*, Trans. Amer. Math. Soc. **131** (1968), 51–78.
23. N. Steenrod, *The topology of fibre bundle*, Princeton Univ. Press, Princeton, N.J., 1951.

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803